A central notion in Functional Analysis

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History, applications and future of super weak compactness

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Plan of the talk

SWC = super weakly compact / super weak compactness

1. History of SWC
2. Definitions, properties and examples
3. Characterizations of SWC
4. Didactical use of uniformly convex functions
5. Applications to Convex Analysis
6. Quantification of SWC
7. Applications to nonseparable theory
8. Miscelany
9. "Take away"

History of SWC and related notions

History of SWC and related notions I

Since the very beginning of Banach space theory, many special properties has been recognised in the spaces L_p with $1 < p < +\infty$: reflexivity, Orlicz's theorem on unconditionally convergent series, uniform convexity... properties going back to the 30's.

The key isomorphic property behind those good features of L_p spaces was discovered in 1972 by R.C. James, who called it *super-reflexivity*. Every uniformly convex space is super-reflexive.

Surprisingly, the same year P. Enflo proved that any super-reflexive space can be renormed to be uniformly convex. That is the most striking example of an isomorphic property characterized by a renorming.

Years later, G. Pisier improved Enflo's result from a quantitative point of view, making so the super-reflexive spaces a suitable infinite-dimensional environment for Analysis, Probability and Approximation Theory.

A remarkable result in nonlinear theory is the metric characterization of super-reflexivity proved by J. Bourgain in 1986.

History of SWC and related notions II

Uniform convexity for functions was introduced by E.S. Levitin and B.T. Polyak in 1966. The theory has been further developed by several authors such as D. Azé and J.P. Penot (1995), or J.M. Borwein, A. Guirao, P. Hájek and V. Vanderwerff (2009).

The operator counterpart of super-reflexivity was introduced by Beauzamy in 1976 under the name of *uniformly convexifying operators*. Lately, those operators were called *super weakly compact* as they can be characterized as a super-property in similar terms to super-reflexive Banach spaces.

It turns out that the set of SWC operators is an *operator ideal* and it is usually denoted as $\mathfrak{W}^{super}$.

A Banach space X is super-reflexive if and only if $\mathbb{I}_X \in \mathfrak{W}^{super}$.

The ideal $\mathfrak{W}^{super}$ lies strictly between $\mathfrak{K}$ and $\mathfrak{W}$. Besides, $\mathfrak{W}^{super}$ is closed and symmetrical, but it lacks the factorisation property.

History of SWC and related notions III

Let me tell you my personal story... That begins when I was a doctoral student with none theorems yet. I met Manuel Cepedello during a conference in London (April 1996), he was student of Gilles Godefroy. Soon after I arrived to Bordeaux (October 1996), Robert Deville told me *"there is an interesting preprint by a spaniard, Manuel Cepedello, do you know him?"*. His result was a nice characterization of super-reflexivity using approximation by differences of convex functions.

Later on, in my Ph.D. thesis (Bordeaux 1998, Murcia 1999) I developed a method for renorming, that I could combine with the ideas of Cepedello to obtain some local results (Israel J. Math. 2003). However, to obtain a uniform property using my method was difficult at that moment, so I managed to adapt Lancien's proof of the Enflo-Pisier theorem to a non-linear setting (JCA 2008).

The result was a theory of finitely dentable maps that applied to real functions generalized Cepedello's results, but applied to the identity of bounded closed convex set provided a localization of super-reflexivity.

History of SWC and related notions IV

My approximation to a local version of super-reflexivity had many issues. Finite dentability was not the right way to recognize a super-property. For instance, finite dentability is not preserved by closed convex hulls. During the following years I turned my attention to different problems in Banach space geometry and topology.

In 2013 I became aware of a paper by Lixin Cheng, Qingjin Cheng, Bo Wang and Wen Zhang (Studia Math. 2010) that contained part of the Ph.D. thesis of Qingjin Cheng. In this paper, it was studied a localization of super-reflexivity from the right point of view, meaning as a super-property. The main results for the convex case had nonempty intersection with mine and their techniques overcome the difficulties I had. Most important, they also coined the term *super weakly compact sets*.

I exchanged some emails with Lixin Cheng and the new ideas from Xiamen gave a boost to my research on SWC (JMAA 2016). Later, I was able to involve in the project to colleagues such as Gilles Lancien and Guillaume Grelier, whose Ph.D. thesis contains a great deal about SWC.

Future of SWC and related notions

Definitions, properties and examples

Definitions (in terms of ultrapowers)

For $\mathcal{U}$ ultrafliter, the *ultrapower* $X^{\mathcal{U}}$ is the quotient of $\ell_\infty(X)$ (bounded sequences from X with the supremum norm) by the subspace of $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n,\mathcal{U}} \|x_n\| = 0$.

A Banach space X is *super-reflexive* if $X^{\mathcal{U}}$ is reflexive for some (or, equivalently, any) free ultrafilter $\mathcal{U}$.

An operator $T : X \to Y$ is SWC (or belongs to the ideal $\mathfrak{W}^{super}$) if the induced operator $T^{\mathcal{U}} : X^{\mathcal{U}} \to Y^{\mathcal{U}}$ is weakly compact for any ultrafilter $\mathcal{U}$ (equivalently, a free ultrafilter on $\mathbb{N}$).

For a set $A \subset X$, let $A^{\mathcal{U}}$ be the subset of $X^{\mathcal{U}}$ whose elements have a representative in $A^{\mathbb{N}}$. A set $A \subset X$ is said relatively SWC if $A^{\mathcal{U}}$ is a relatively weakly compact subset of $X^{\mathcal{U}}$ for some (or, equivalently, any) free ultrafilter $\mathcal{U}$.

Finally, a SWC set is a weakly compact set that is also relatively SWC.

A few facts

Some properties linking those notions among them:

1. The class of SWC sets strictly lies between the classes of the compact and the weakly compact sets.
2. The Banach space X is super-reflexive if and only if B_X is SWC.
3. $T : X \to Y$ is in $\mathfrak{W}^{super}$ if and only if $T(B_X)$ is relatively SWC.
4. If $K \subset X$ is SWC then there is a reflexive space Z and $T \in \mathfrak{W}^{super}(Z, X)$ such that $K \subset T(B_Z)$.
5. There is a SWC set that does not embed linearly isomorphically into a super-reflexive Banach space.
6. That implies that $\mathfrak{W}^{super}$ does not have the factorization property.

A few more facts

The properties that we could expect...and two peculiarities.

1. The class of SWC sets behaves quite well by usual vector topology operations: closed subsets, finite products, algebraic sums, linear continuous images...of sets from the class remain in the class.
2. Closed convex hulls of SWC are SWC (K. Tu, 2021).
3. Given $A \subset X$, if for every $\varepsilon > 0$ there is SWC set $A_\varepsilon \subset X$ such that $A \subset A_\varepsilon + \varepsilon B_X$, then A is relatively SWC.
4. SWC is not characterized by sequences.
5. SWC is not characterized by the metric: there are two successions in c_0 with weak limit zero and distance one between elements, being one of them relatively SWC and the other not the other. The way the set is put into the Banach space is crucial.
6. The relation between SWC and uniform Eberlein will be detailed later.

SWC sets are everywhere

Super weak compactness is more spreaded than it may appear.

1. Every weakly compact subset of a super-reflexive space is SWC.
2. Every weakly compact subset of $L_1(\mu)$ is SWC.
3. In particular, every weakly compact operator to an $L_1(\mu)$ space is SWC.
4. Every weakly compact subset of $C(K)^*$ is SWC.
5. Every weakly compact operator defined on a $C(K)$ space is SWC.
6. In particular, any operator from $C(K)$ to a Banach space that not contains c_0 is SWC.
7. Every weakly compact subset of a JBW*-triple predual is SWC (in particular, that applies to preduals of von Neumann algebras).

SWC sets are everywhere

One last example: *uniformly weakly null sets* are SWC:

We say that $A \subset X$ is uniformly weakly null if for every $\varepsilon > 0$ such that there is $n \in \mathbb{N}$ such that for every $x^* \in B_{X^*}$ then

$$|\{x \in A : |x^*(x)| > \varepsilon\}| \leq n.$$

Analogously, it is possible to define uniformly weakly convergent sequences. That notion appears in the study of the Banach-Saks property.

Important example of uniformly weakly null sets come from Schauder bases: if $\{e_i : i \in I\}$ is a (long) Schauder basis of Banach space X not isomorphic to $\ell_1(I)$, then $\{e_i : i \in I\} \cup \{0\}$ is SWC.

An example: the SWC sets of c_0

The standard basis of c_0 is relatively SWC. Indeed, it is a uniformly weakly null set, but it is easier to realize that it is covered by the injection of ℓ_2 given by the identity. Analogously, any bounded subset of c_0 made of finitely supported vectors and such that the cardinal of the supports is uniformly bounded is relatively SWC.

Now consider subsets of c_0 which are families of characteristic functions of finite sets of $\mathbb{N}$, namely sets $K = \{\chi_F : F \in \mathcal{F}\}$ for $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$.

Lancien, R. (2022)

Let $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ a family of subsets such that $K = \{\chi_F : F \in \mathcal{F}\}$ is a SWC subset of c_0. Then, there exists $p \in \mathbb{N}$ and $C > 0$ such that

$$\forall A \in [\mathbb{N}]^{<\omega}, \quad \left|\{F \cap A : F \in \mathcal{F}\}\right| \leq C|A|^p.$$

The *Schreier family* $\mathcal{S}$ is made up of those $F \in [\mathbb{N}]^{\omega}$ such that $|F| \leq \min(F)$. The set $\{\chi_F : F \in \mathcal{S}\}$ is weakly compact in c_0 however

$$\left|\{F \cap [1, N] : F \in \mathcal{S}\}\right| > 2^{\frac{N}{2}-1}.$$

Characterizations of SWC

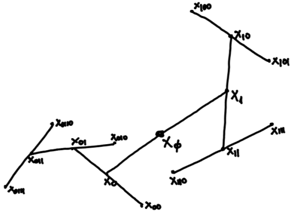
Some notions we need I

The following notions are related to characterizations of super-reflexivity.

- *Uniformly convex functions*: A proper convex function f defined on a Banach space X is said uniformly convex if for every $\varepsilon > 0$ there is $\delta > 0$ such that
$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} - \delta.$$
whenever $\|x-y\| \geq \varepsilon$.
- *Banach-Saks property*: If $K \subset X$ is SWC and $(x_n) \subset K$ is an infinite sequence then there is a subsequence (x_{n_k}) whose Cesàro means, that is
$$\frac{x_{n_1} + x_{n_2} + \cdots + x_{n_k}}{k}$$
converge in norm to some $x \in K$.

Some notions we need II

- *Dyadic trees*. A dyadic tree of height $n \in \mathbb{N}$ is a set of the form $\{x_s : |s| \leq n\}$, indexed by finite sequences $s \in \bigcup_{k=0}^{n} \{0,1\}^k$ of length $|s| \leq n$, such that
$$x_s = 2^{-1}(x_{s \frown 0} + x_{s \frown 1})$$
for every $|s| < n$, where $\{0,1\}^0 := \{\emptyset\}$ indexes the root $x_\emptyset$ and the symbol "$\frown$" stands for concatenation.
We say that a dyadic tree $\{x_s : |s| \leq n\}$ is ε*-separated* if for every $|s| < n$
$$\|x_{s \frown 0} - x_{s \frown 1}\| \geq \varepsilon.$$



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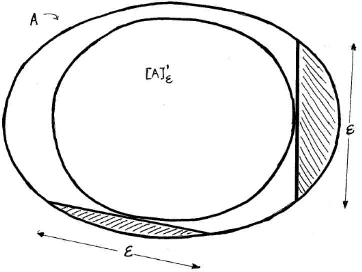
$$\|x_{s\frown 0} - x_{s\frown 1}\| \geq \varepsilon.$$

- *d.c.-Lipschitz functions*.Let $C \subset X$ be convex closed. A function $f : C \to \mathbb{R}$ is said d.c.-Lipschitz if it is the difference of two bounded convex Lipschitz functions.

Some notions we need III

- *Finite dentability*. Consider $A \subset X$ and denote by $\mathbb{H}$ the family of open halfspaces.
    1. $[A]'_\varepsilon = A \setminus \bigcup \{H \in \mathbb{H} : \mathrm{diam}(A \cap H) < \varepsilon\}$.
    2. $[A]^{n+1}_\varepsilon = [[A]^n_\varepsilon]'_\varepsilon$.



Some notions we need III

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  2. $[A]^{n+1}_\varepsilon = [[A]^n_\varepsilon]'_\varepsilon$.
  3. $Dz(A, \varepsilon) = \inf\{n : [A]^n_\varepsilon = \emptyset\}$

  We say that a set $A \subset X$ is *finitely dentable* if $Dz(A, \varepsilon) < \omega$ for every $\varepsilon > 0$.

All the previous notions are usually related to super-reflexivity.

The following follows from a remarkable result of Pisier: if X is a super-reflexive Banach space, then there exists $p \geq 2$ and $c > 0$ such that

$$Dz(B_X, \varepsilon) \leq c\varepsilon^{-p}.$$

Characterizations of SWC convex sets

R. (2008), Cheng, Cheng, Wang, Zhang (2010), Lancien, R. (2022)

For a closed convex bounded set $C \subset X$, the following statements are equivalent:

1. C is SWC;
2. There exists a Banach space Z and a SWC operator $T : Z \to X$ such that $C \subset T(B_Z)$;
3. For every $\varepsilon > 0$, C does not contain arbitrarily large ε-separated dyadic trees;
4. C is finitely dentable;
5. C supports a bounded uniformly convex function;
6. There is an equivalent norm $||| \cdot |||$ on X such that $||| \cdot |||^2$ is uniformly convex on C;
7. Every Lipschitz function on C can be uniformly approximated by d.c.-Lipschitz functions;

More properties of SWC sets

R. (2008), Cheng, Cheng, Luo, Tu, Zhang (2018), Lancien, R. (2022)

Let $K \subset X$ be a SCW set. Then:

(a) K is uniformly Eberlein endowed with the weak topology;

(b) K has the Banach-Saks property;

(c) K is finitely dentable.

None of the above properties can be reversed.

In relation with nonlinear theory, unfortunately SWC sets cannot be metrically characterized in general. Being SWC depends on how the set is placed into the Banach spaces.

For SWC convex sets it is possible an almost-nonlinear characterization, which is similar to the bimetric characterization of SWC operators given by Causey and Dilworth.

Didactical use of uniformly convex functions

Convexity for students

When solving optimization problems in an infinite dimensional frame compactness is seldom available. Instead, we could get profit from convexity (and completeness).

For instance, a convex lower semicontinuous function can be perturbed by a continuous afine function in order to reach its minimum (Bishop-Phelps).

A local minimum for a convex function is global. That is guarantied by the annihilation of the differential when available. We recommend the interesting book “Convex Functions” by Borwein and Vanderwerff for a discussion and the application to variational calculus.

A proper lower semicontinuous uniformly convex function is bounded below, strongly coercive and attains a strong minimum (see Zălinescu's book “Convex Analysis in General Vector Spaces”).

Didactical use of uniformly convex functions

The squared norm in a Hilbert space is a uniformly convex function. Indeed,

$$\frac{\|x\|^2 + \|y\|^2}{2} - \|\frac{x+y}{2}\|^2 = \|\frac{x-y}{2}\|^2.$$

Let H be a real Hilbert space, $\phi : H \to \mathbb{R}$ a continuous linear functional. Then the function defined by

$$f(x) = \frac{1}{2}\|x\|^2 - \phi(x) = \frac{1}{2}\langle x, x\rangle - \phi(x)$$

is a continuous uniformly convex function. Let $z \in H$ the point where f attains its minimum, since f is differentiable at z we have

$$0 = df(z)(x) = \langle x, z\rangle - \phi(x)$$

for all $x \in H$. That is the Riesz-Fréchet representation theorem (the complex case can be also deduced with some minor modifications).

Applications to Convex Analysis

Approximation by d.c.-functions

Recall that this story, for me, started with Cepedello's characterization of super-reflexivity.

The definition of finitely dentable can be stated for maps with values in a metric space and, in particular, for real functions.

Grelier, R (2022)

Let $C \subset X$ be a bounded closed convex set and let $f : C \to \mathbb{R}$ be a uniformly continuous function. Consider the following numbers:

(ε_1) the infimum of the $\varepsilon > 0$ such that $\mathrm{Dz}(f, \varepsilon) < \omega$;

(ε_2) the infimum of the $\varepsilon > 0$ such that there exists a d.c.-Lipschitz function g such that $\|f - g\|_C < \varepsilon$.

Then $\varepsilon_1/2 \leq \varepsilon_2 \leq 2\varepsilon_1$.

Approximation by d.c.-functions

We need to recall some definitions. A subset $D \subset X$ is said to be a $(\mathcal{C} \setminus \mathcal{C})_\sigma$-*set* if $D = \bigcup_{n=1}^{\infty}(A_n \setminus B_n)$, where A_n and B_n are convex closed subsets of X. A real function $f\colon C \to \mathbb{R}$ is said to be $(\mathcal{C} \setminus \mathcal{C})_\sigma$-*measurable* if the sets $f^{-1}(-\infty, r)$ and $f^{-1}(r, +\infty)$ are both $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subsets of X for each $r \in \mathbb{R}$.

Garcia-Lirola, R (2017)

Let $f\colon C \to \mathbb{R}$ be a uniformly continuous function...

(i) f is uniform limit of d.c.-bounded functions;

(ii) f is finitely dentable;

(iii) f is uniform limit of d.c.-Lipschitz functions;

(iv) f is countably dentable;

(v) f is $(\mathcal{C} \setminus \mathcal{C})_\sigma$-measurable;

(vi) f is pointwise limit of d.c.-Lipschitz functions.

Then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \not\Rightarrow (v) \not\Rightarrow (iv) \not\Rightarrow (iii)$.

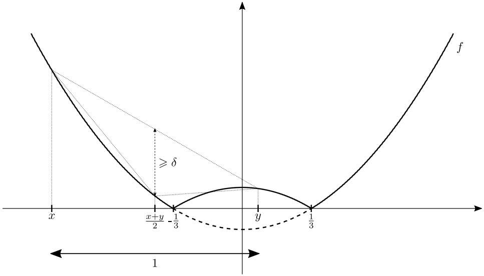
Discrete uniform convexity

Let $\varepsilon > 0$. A proper function f defined on a Banach space X is said *ε-uniformly convex* if there is $\delta > 0$ such that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} - \delta$$

whenever $\|x-y\| \geq \varepsilon$.

Note that an ε-uniformly convex function does not need to be convex.



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whenever $\|x-y\| \geq \varepsilon$.

Note that an ε-uniformly convex function does not need to convex.

Let $\varepsilon > 0$. A proper function f defined on a Banach space X is said *ε-uniformly quasi-convex* if there is there is some $\delta > 0$ such that

$$f\left(\frac{x+y}{2}\right) \leq \max\{f(x), f(y)\} - \delta$$

whenever $x, y \in X$ with $\|x-y\| \geq \varepsilon$.

Discrete uniform convexity

Fact 1: Let $C \subset X$ be a convex set. Then the function defined by

$$f(x) = -\sup\{\text{height}(x_s) : (x_s) \subset C \ \varepsilon\text{-sep. tree}, x_\emptyset = x\}$$

and $f(x) = +\infty$ if $x \notin C$, is ε-uniformly quasi-convex.

Fact 2: Let $\varepsilon > 0$ and let $f : X \to \mathbb{R}$ be a bounded below ε-uniformly quasi-convex function with modulus $\delta > 0$. Then the function $h \circ f$ is ε-uniformly convex, where $h(t) = 3^{t/\delta}$.

Grelier, R. (2022)

Let $C \subset X$ be a convex set. The following are equivalent:

1. $C \subset X$ does not admit arbitrarily large ε-separated dyadic trees;
2. $C \subset X$ supports an ε-uniformly convex bounded function.

Discrete uniform convexity

Let $\operatorname{conv} f$ denote the lower convex envelope of a function f.

Grelier, R. (2022)

Let $\varepsilon > 0$ and let f be an ε-uniformly convex function. Then the following statements are equivalent:

1. $\operatorname{conv} f$ is proper;
2. f is bounded below;
3. f is bounded below by an affine continuous function.

In case one of them holds, then $\operatorname{conv} f$ will be ε^+-uniformly convex.

Corollary

Let C be a set that supports a bounded ε-uniformly convex function for every $\varepsilon > 0$. Then C supports a bounded uniformly convex function.

Enflo's renorming result can be retrieved with these ideas.

Quantification of SWC

Quantification of SWC

The most important difference between super-reflexivity and SWC is that the last one admits quantification. The same can be said of reflexivity and weakly compactness.

For $A \subset X$ we will consider the measure of weak non-compactness introduced by M. Fabian, P. Hájek, V. Montesinos and V. Zizler, defined by the formula

$$\gamma(A) = \inf\{\varepsilon > 0 : \overline{A}^{w^*} \subset X + \varepsilon B_{X^{**}}\}.$$

The corresponding "super" definition (Grelier, R.) should be

$$\Gamma(A) := \gamma(A^{\mathcal{U}})$$

Apparently, the definition of Γ depends on the choice of the ultrafilter $\mathcal{U}$.

Quantification of SWC

Grelier, R. (2022)

Let $C \subset X$ be a bounded closed convex subset. Consider the following numbers:

(η_1) $= \Gamma(C)$;

(η_2) the supremum of the numbers $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there are $x_1, \dots, x_n \in C$ such that $\mathrm{d}(\mathrm{conv}\{x_1, \dots, x_k\}, \mathrm{conv}\{x_{k+1}, \dots, x_n\}) \geq \varepsilon$ for all $k = 1, \dots, n-1$;

(η_3) the supremum of the $\varepsilon > 0$ such that there are ε-separated dyadic trees in C of arbitrary height;

(η_4) $= \Delta(C^{\mathcal{U}})$ (measure of dentability);

(η_5) the infimum of the $\varepsilon > 0$ such that $\mathrm{Dz}(C, \varepsilon) < \omega$;

(η_6) the infimum of the $\varepsilon > 0$ such that C supports a convex bounded ε-uniformly convex function.

Then $\eta_1 \leq \eta_2 \leq 2\eta_3 \leq 2\eta_4 \leq 2\eta_1$ and $\eta_4 \leq 2\eta_5 \leq 2\eta_6 \leq 2\eta_2$.

Applications to non-separable theory

Spaces generated by ideals, sets or. . . other spaces

A Banach space X is said to be *generated* by a class of sets $\mathfrak{S}$, or a class of operators $\mathfrak{O}$, or a class of Banach spaces $\mathfrak{B}$ if, respectively,

1. there is $A \in \mathfrak{S}$ such that $\mathrm{span}(A)$ is dense in X;
2. or there is $T \in \mathfrak{O}$ with $T : Z \to X$ such that $T(Z)$ is dense;
3. or there is $Z \in \mathfrak{B}$ with $T : Z \to X$ such that $T(Z)$ is dense.

As class of operator we will always take an operator ideal. In the case of $\mathfrak{W}$ is more interesting, as showed by this consequence of the celebrated result of Davis, Figiel, Johnson and Pelczynski.

For a Banach space X the following are equivalent:

(a) There is $K \subset X$ weakly compact with $\overline{\mathrm{span}}(K) = X$;

(b) there are Z and $T \in \mathfrak{W}(Z, X)$ such that $\overline{T(Z)} = X$;

(c) there are a reflexive space Z and $T \in \mathcal{L}(Z, X)$ such that $\overline{T(Z)} = X$.

The spaces enjoying those properties are calle *weakly compactly generated* and the class is denoted WCG.

Some geometric properties of norms

The following definitions are needed too...

Given a Banach space $(X, \|\cdot\|)$:

1. The norm is *uniformly Gâteaux* (UG) smooth if for every $h \in X$
$$\sup\{\|x+th\| + \|x-th\| - 2 : x \in S_X\} = o(t) \text{ when } t \to 0.$$
2. Given a bounded set $H \subset X$, the norm is H-UG smooth if
$$\sup\{\|x+th\| + \|x-th\| - 2 : x \in S_X, h \in H\} = o(t) \text{ when } t \to 0.$$
3. The norm is *uniformly Fréchet* smooth if it is B_X-UG.
4. The norm is *strongly* UG smooth if it is H-UG smooth for some bounded and linearly dense subset $H \subset X$.

We say that X is *strongly generated* by $A \subset X$ if for every $K \subset X$ weakly compact and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $K \subset nA + \varepsilon B_X$. This notion applies to the other kinds of generation.

Generation by several kinds of spaces

Pay attention to the following result

Fabian, Godefroy, Hájek, Zizler (2003)

For a Banach space X consider the assertions:

(i) X is Hilbert-generated.

(ii) X is superreflexive-generated.

(iii) X is generated by the ℓ_2-sum of superreflexive spaces.

(iv) X admits an equivalent strongly UG smooth norm.

(v) X is WCG and admits an equivalent UG smooth norm.

(vi) X is a subspace of a Hilbert-generated space.

Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi)$. Moreover, no one of these implications can be reversed in general.

Question: Which of these statements does not fit with the others?
Answer: **(iv)**, because it does not contain the word "generated".

Renorming characterisation of super WCG

But the class in statement (iv) is actually generated too

R. (2016)

A Banach space X is generated by a SWC set (super WCG) if and only if it admits an equivalent strongly UG smooth norm.

Moreover, if we consider strong generation (weakly compact subsets are somehow "captured" by the generating set) we have have a remarkable renorming property.

R. (2016)

Let X be a Banach space strongly generated by a SWC set (strongly super WCG). Then there is an equivalent norm on X such that its restriction to any reflexive subspace is both uniformly convex and uniformly Fréchet smooth.

That includes, in particular, $L_1(\mu)$ spaces.

Subspaces of WCG and Hilbert generated spaces

Moreover, with the help of the measures of non-compactness we can add an internal property to the classic characterizations.

Benyamini, Rudin, Wage (1977), Fabian, Montesinos, Zizler (2004)

For a Banach space X the following statements are equivalent:

(i) X is a subspace of a WCG space;

(ii) (B_{X^*}, w^*) is an Eberlein compact;

(iii) For every $\varepsilon > 0$ there are sets (A_n^ε) such that $B_X = \bigcup_{n=1}^\infty A_n^\varepsilon$ and $\gamma(A_n^\varepsilon) < \varepsilon$.

Benyamini, Rudin, Wage (1977), Grelier, R. (2022)

For a Banach space X the following statements are equivalent:

(i) X is a subspace of a Hilbert generated space;

(ii) (B_{X^*}, w^*) is a uniform Eberlein compact;

(iii) For every $\varepsilon > 0$ there are sets (A_n^ε) such that $B_X = \bigcup_{n=1}^\infty A_n^\varepsilon$ and $\Gamma(A_n^\varepsilon) < \varepsilon$.

Miscelany

Back to uniformly convex spaces

Modulus of uniform convexity

$$\delta_X(t) = \inf\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1, \|x-y\| \geq t\}$$

Pisier proved that is posible to renorm a super-reflexive space X in such a way that $\delta_{\|\cdot\|}(t) \geq ct^p$ for some $p \geq 2$.

Consider the following order for real functions defined on $(0,1]$. We say that $\phi \preceq \psi$ if there is a constant $c > 0$ such that $\phi(t) \leq c\,\psi(t)$ for all $t \in (0,1]$. If $\phi \preceq \psi$ and $\psi \preceq \phi$, then we say that ϕ and ψ are *equivalent*.

García-Lirola, R. (2021)

Let X be a superreflexive Banach space. Then the supremum with respect to the order $\preceq$ of the set

$$\{\delta_{|||\cdot|||}(t) : |||\cdot||| \text{ is an equivalent norm on } X\}$$

is equivalent, in the same order, to $Sz(B_{L^2(X)}, t)^{-1}$ $(\sim Dz(B_{L^2(X)}, t)^{-1})$.

Fixed Point Property

Convex SWC sets have normal structure under suitable renormings, so they have the fixed point property (FPP) for non-expansive mappings. However, the basic example of convex weakly compact not having the FPP for a non-linear isometry is SWC. That forces us to consider stronger hypothesis on the self mappings, such as to be linear or affine.

Cheng, Cheng, Zhang (2015)

Suppose that C is a nonempty closed bounded convex subset of a Banach space X. Then the following statements are equivalent.

(i) C is SWC;

(ii) C has the super FPP for linear isometries;

(iii) C has the super FPP for affine isometries.

Ergodic Theory

Let C be a convex subset of a Banach space X. We say that the mapping $T : C \to C$ is *ergodic* if the Cesàro mean sequence

$$\frac{1}{n}\sum_{k=0}^{n-1} T^k(x)$$

converges for every $x \in C$. We say that C is ergodic itself if any affine $T : C \to C$ that makes the previous sequence equicontinuous is ergodic.

Grelier, R. (2024)

Let C be a bounded closed convex subset of a Banach space X. The following assertions are equivalent:

(i) C is SWC;

(ii) C is super-ergodic;

(iii) any affine isometry from a closed convex set f.r. in C into itself is ergodic.

Non-linear Functional Analysis

The authoritative book Geometrical Nonlinear Functional Analysis by Benyamini and Lindenstrauss begins with the study of absolute uniform retracts.

Cheng, Cheng, Whang (2016)

Every super weakly compact convex subset of a Banach space is an absolute uniform retract.

Lipschitz-free spaces is an active research area where SWC sets have found a place.

Silber (2024)

Let M be a (norm) compact subset of a superreflexive space X. Then every relatively weakly compact of the Lipschitz-free space $\mathcal{F}(M)$ is relatively SWC.

There is a more recent development by R. Aliaga, E. Pernecká and A. Quero.

Take away (some problems)

Take away

1. Characterize the subsets of a Banach space that embeds into a super-reflexive space.
2. Is any super WCG space generated by a uniformly weakly null set?
3. Build the theory of asymptotic SWC. . . Is it interesting?
4. What is the role of uniform smoothness?
5. Prove a factorization theorem for SWC operators with a controlled dentability index.
6. Is a weakly compact with Szlenk index at most ω uniformly Eberlein?
7. Produce a more topological theory of super compactness.

非常感謝您的關注！

All my papers are available here

https://webs.um.es/matias (click on “Investigación”)