Area, a notion not as simple as it seems

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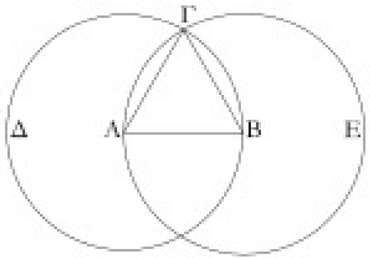
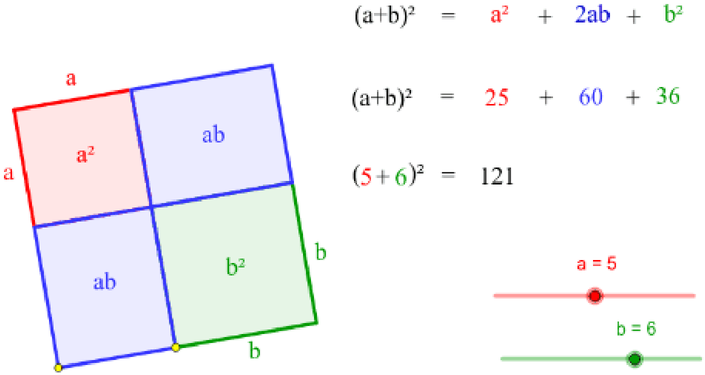
Geometric intuition

Fact: geometrical intuition is a powerful didactic resource.



But how much can we use (or abuse) geometric images without exposing the rigor? As rigorous thinking should be one of the skills provided by the early study of mathematics. The key is not to confuse intuition (a plausible statement) with truth (those statements that are already proved).

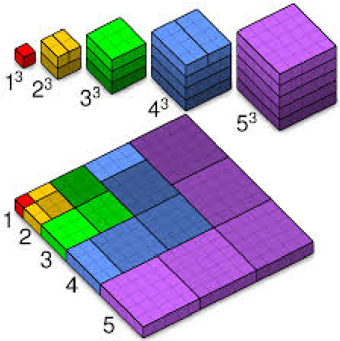
Arithmetic and Geometry



That is a geometric illustration of an arithmetic identity.

Arithmetic and Geometry

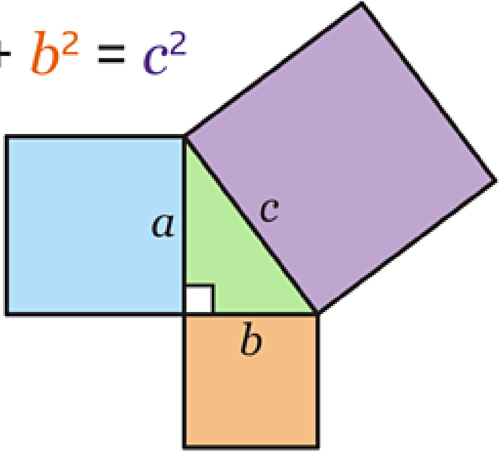
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



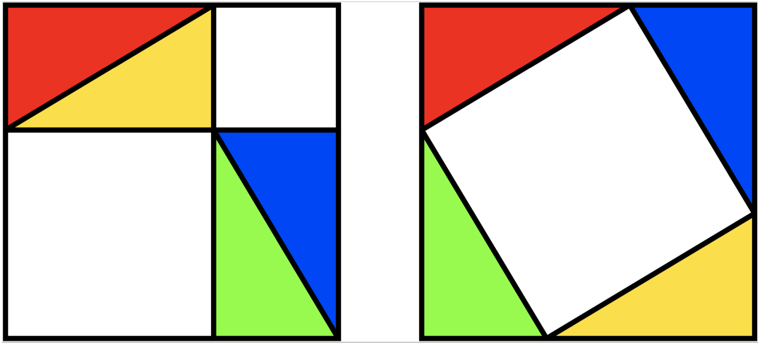
That is even better!

Pythagoras Theorem

$a^2 + b^2 = c^2$



A proof of Pythagoras Theorem?



Is that a mathematical proof?

The proof mathematicians like... why?

I.47. *En un triángulo rectángulo el cuadrado de la hipotenusa es igual a la suma de los cuadrados de los dos catetos.*

En el triángulo ABC, rectángulo en C, trácese CF, perpendicular a la hipotenusa AB, como se ha hecho en la figura 1.3*d*. Tenemos así tres triángulos rectángulos semejantes, ABC, ACF, CBF, cuyas hipotenusas son AB, AC, CB. Por VI.19, sus áreas satisfacen la proporción

$$\frac{ABC}{AB^2} = \frac{ACF}{AC^2} = \frac{CBF}{CB^2}.$$

Es evidente que $ABC = ACF + CBF$. Por lo tanto, $AB^2 = AC^2 + CB^2$.

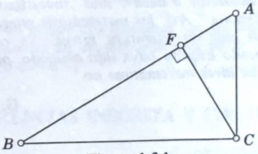


Figura 1.3d

Conservation of mass: an intuition

We learn in childhood that the sum of the weight of parts coincide with the weight of the total.

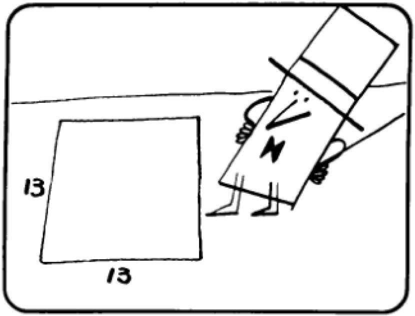
Conservation of mass: a scientific fact

Moreover, the mass is conserved not only for division into pieces. Lavoisier proved that the mass is conserved in chemical reactions.

Actually, when a great amount of energy is involved is necessary to use Einstein's equivalence formula $E = mc^2$ to keep a right balance.

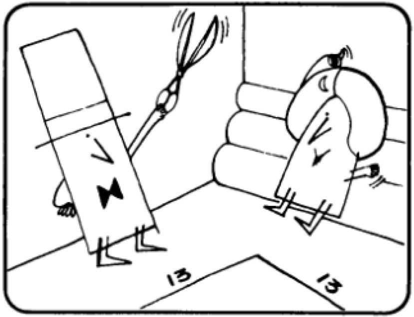
Area, or volume, is not the same that mass

Are we sure that the area is invariant by decompositions into pieces? Pay attention to the following stroy.



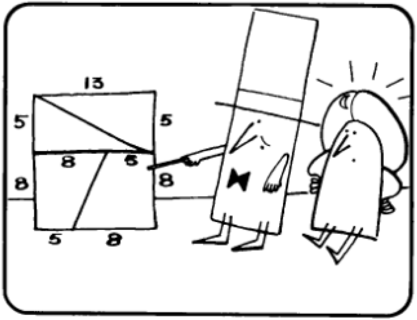
Mr. Randi, the world famous magician, owns a rug that is 13 decimeters by 13 decimeters. He wants to change it to an 8-by-21 rug. Mr. Randi took the rug to Omar, a rug dealer.

Area, or volume, is not the same that mass



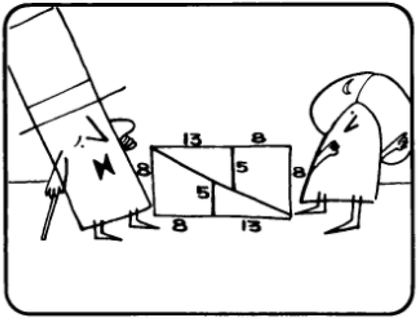
Randi: Omar, my friend, I want you to cut this rug into four pieces, then sew them together to make an 8-by-21 rug.
Omar: I'm sorry, Mr. Randi. You're a great magician but your arithmetic is terrible: 13-by-13 is 169, 8-by-21 is 168. It won't work.

Area, or volume, is not the same that mass



Randi: My dear Omar. The great Randi is *never* wrong. Kindly cut the rug into four pieces like this.

Area, or volume, is not the same that mass



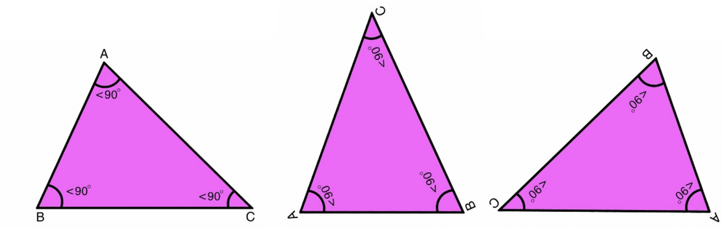
Omar did as he was told. Then Mr. Randi arranged the pieces, and Omar sewed them together to make an 8-by-21 rug. **Omar:** I can't believe it! The area has shrunk from 169 to 168! What happened to that missing square decimeter?

Is there anything weird? Think carefully. . . We trust so much on the invariance of area that Randi cannot cheat us.

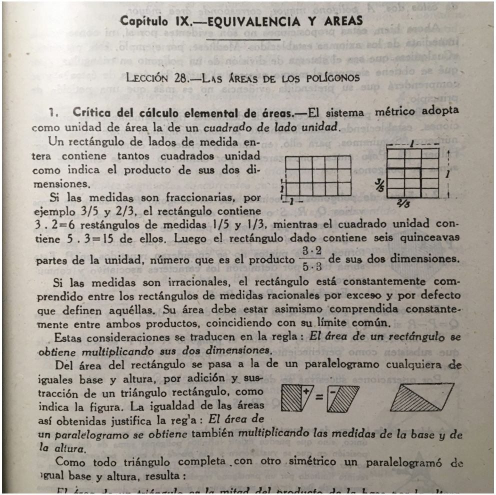
The foundation of a mathematical theory of magnitude

So far we have think about area as preexisting notion. Even, when we study the notion of integral for the first time the teacher says to us that the integral represents the area under the graph. At the end, we can recover a definition of area from integration theory but that is a "privilege" reserved for mathematicians. . . The challenge is how to define rigorously a notion of area at an elementary level.

If we wish to provide a rigorous definition of area for polygons, we should start by the triangles. . . and the first difficulties show up.



The foundation of a mathematical theory of magnitude



Capítulo IX.—EQUIVALENCIA Y AREAS

LECCIÓN 28.—LAS ÁREAS DE LOS POLÍGONOS

1. **Crítica del cálculo elemental de áreas.**—El sistema métrico adopta como unidad de área la de un *cuadrado de lado unidad*.

Un rectángulo de lados de medida entera contiene tantos cuadrados unidad como indica el producto de sus dos dimensiones.

Si las medidas son fraccionarias, por ejemplo 3/5 y 2/3, el rectángulo contiene 3 . 2=6 restángulos de medidas 1/5 y 1/3, mientras el cuadrado unidad contiene 5 . 3=15 de ellos. Luego el rectángulo dado contiene seis quinceavas partes de la unidad, número que es el producto $\frac{3\cdot 2}{5\cdot 3}$ de sus dos dimensiones.

Si las medidas son irracionales, el rectángulo está constantemente comprendido entre los rectángulos de medidas racionales por exceso y por defecto que definen aquéllas. Su área debe estar asimismo comprendida constantemente entre ambos productos, coincidiendo con su límite común.

Estas consideraciones se traducen en la regla: *El área de un rectángulo se obtiene multiplicando sus dos dimensiones.*

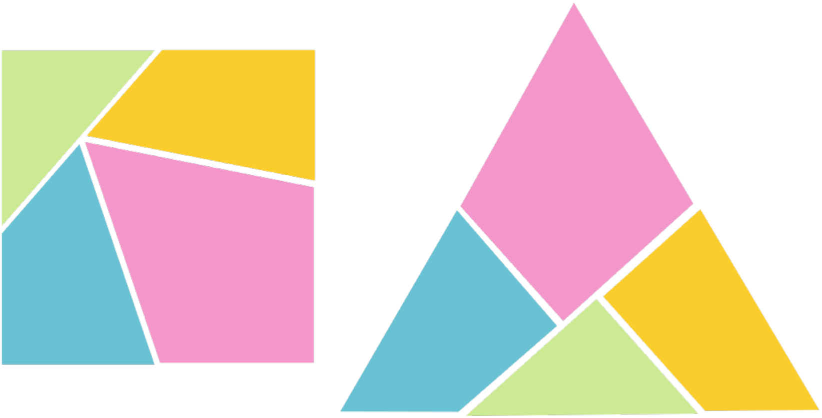
Del área del rectángulo se pasa a la de un paralelogramo cualquiera de iguales base y altura, por adición y sustracción de un triángulo rectángulo, como indica la figura. La igualdad de las áreas así obtenidas justifica la reg'a: *El área de un paralelogramo se obtiene* también *multiplicando las medidas de la base y de la altura.*

Como todo triángulo completa con otro simétrico un paralelogramó de igual base y altura, resulta:

El área de un triángulo es la mitad del producto de la base por la altura

Once we have the fundaments of our theory, we can explore related problems.

Theorem of Bolyai-Gerwien



Two polygons with the same are can decomposed into the same polygonal pieces.

Is there a theorem of Bolyai-Gerwien for polyhedra?

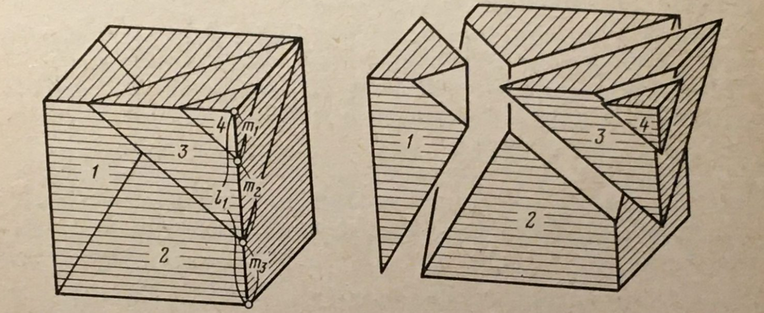
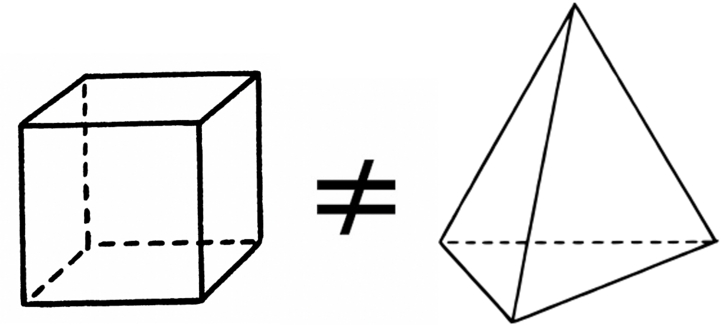


Fig. 49

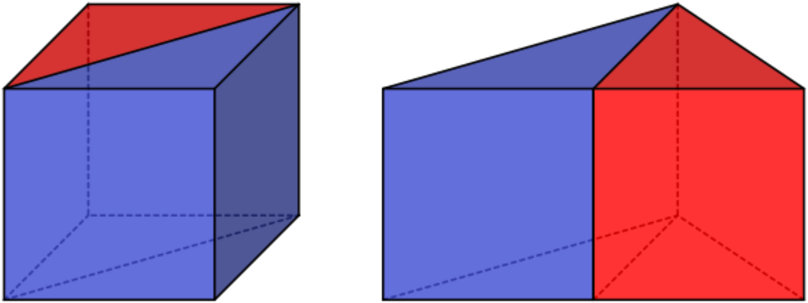
Can be a cube decomposed into smaller polyhedra in order to produce a tetrahedron? That was asked by David Hilbert among the famous 23 problems list he posed in 1900.

Is there a theorem of Bolyai-Gerwien for polyhedra? NO



Dehn proved that cube and tetrahedron of the same volume are not equivalent by decomposition into smaller polyhedra.

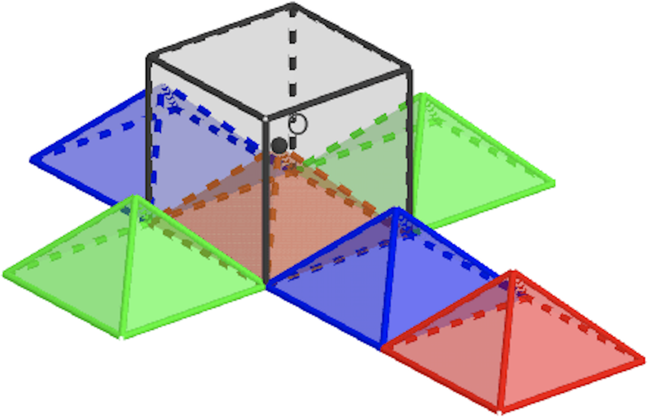
Dehn's invariant



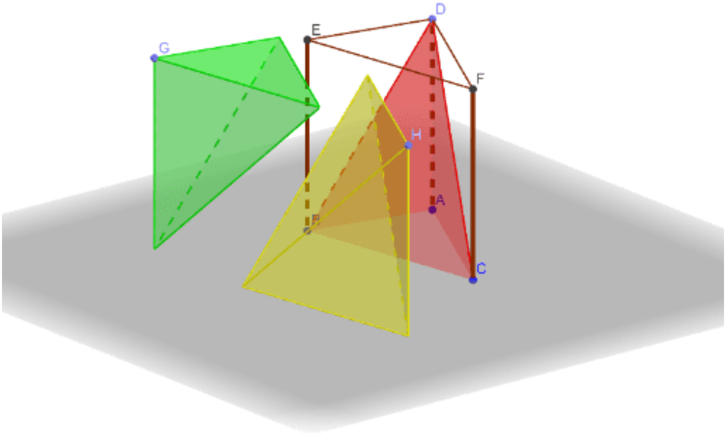
Dehn found a magnitude associated to polyhedra that is invariant by rearrangements and proved its value is different for cube and tetrahedron.

The volume of the pyramid

How we can justify the formula for the volume of a pyramid at an elementary level?

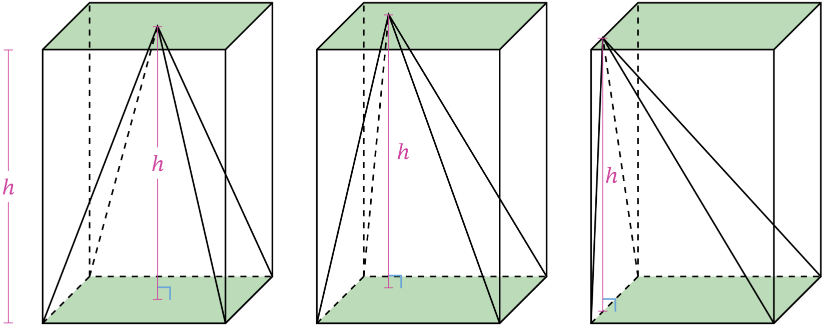


The volume of the pyramid



The decomposition of a triangular prisma into three pyramids with the same volume.

The volume of the pyramid: Cavalieri



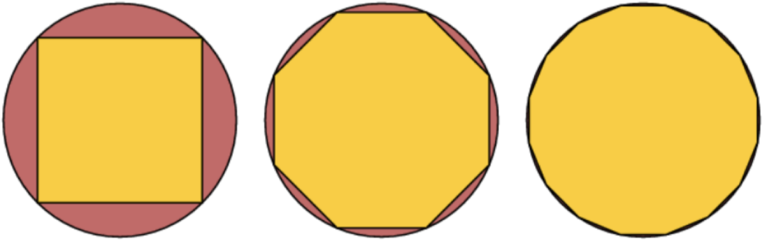
The idea is to show that two pyramids with equal base and height must have the same volume.

Cavalieri

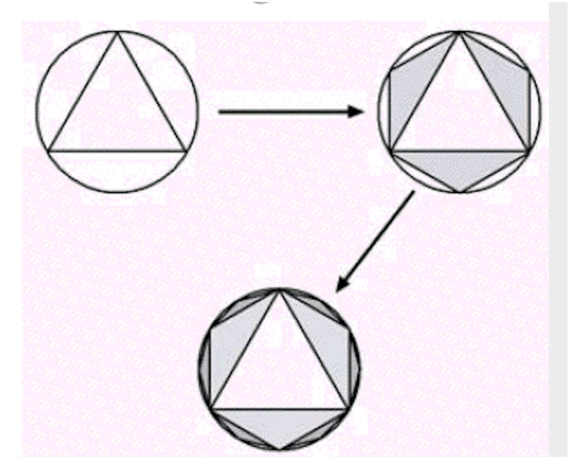
Isn't Cavalieri's principle rather a geometric intuition that a fact?

Area of circle

Cavalieri's principle can be justified by approximation... but that implicitly assumes that the volume of both sets exists a priori. We find the same obstacle with the area of the circle.



The circle as a countable union of triangles



Exhaustion method can be understand in terms of an infinite decomposition... in the case of the circle we can use triangles.

Series

TABLA DE SUMAS DE ALGUNAS SERIES NUMÉRICAS:

1) $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!}+\ldots=e.$

2) $1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots\pm\frac{1}{n!}\mp\ldots=\frac{1}{e},$

3) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\pm\frac{1}{n}\mp\ldots=\ln 2,$

4) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}+\ldots=2,$

5) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots\pm\frac{1}{2^n}\mp\ldots=\frac{2}{3},$

6) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\pm\frac{1}{2n-1}\mp\ldots=\frac{\pi}{4},$

7) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}+\ldots=1,$

8) $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}+\ldots=\frac{1}{2},$

9) $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(n-1)(n+1)}+\ldots=\frac{3}{4},$

10) $\frac{1}{3\cdot 5}+\frac{1}{7\cdot 9}+\frac{1}{11\cdot 13}+\ldots+\frac{1}{(4n-1)(4n+1)}+\ldots=\frac{1}{2}-\frac{\pi}{8},$

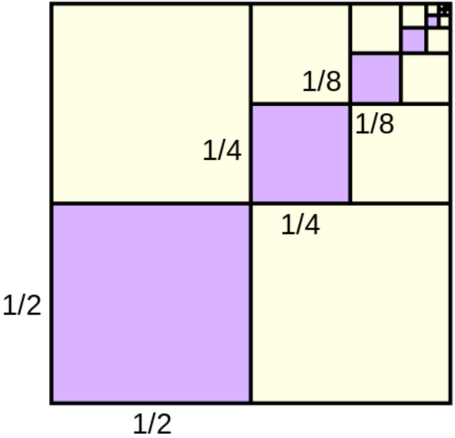
11) $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{n(n+1)(n+2)}+\ldots=\frac{1}{4},$

12) $\frac{1}{1\cdot 2\ldots l}+\frac{1}{2\cdot 3\ldots(l+1)}+\ldots+\frac{1}{n\ldots(n+l-1)}+\ldots=\frac{1}{(l-1)(l-1)!},$

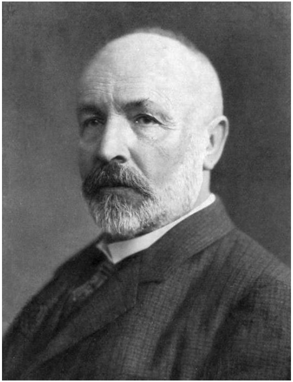
13) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}+\ldots=\frac{\pi^2}{6},$

Series

$$\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\frac{1}{4^4}+\frac{1}{4^5}+\cdots = ?$$



Georges Cantor



$\mathbb{N}$	$\leftrightarrow$	*reals in* (0,1)
1	$\leftrightarrow$	.835987...
2	$\leftrightarrow$	.250000...
3	$\leftrightarrow$	.559423...
4	$\leftrightarrow$	.500000...
5	$\leftrightarrow$	.728532...
6	$\leftrightarrow$	.845312...
$\vdots$		$\vdots$
n	$\leftrightarrow$	$.r_1 r_2 r_3 r_4 r_5 \ldots r_n \ldots$
$\vdots$		$\vdots$

Dealing with the infinite

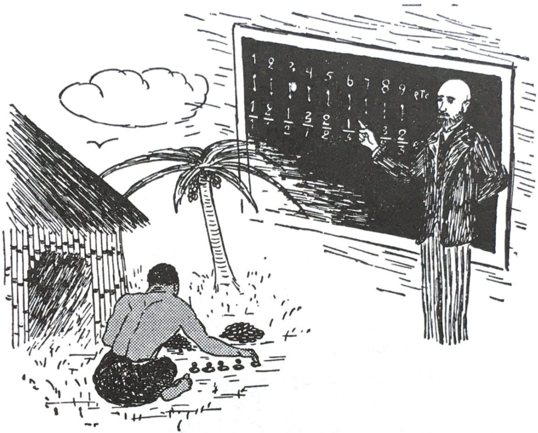
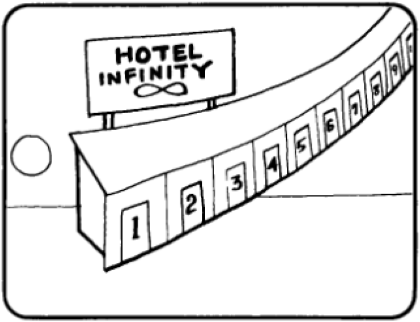


FIGURE 5

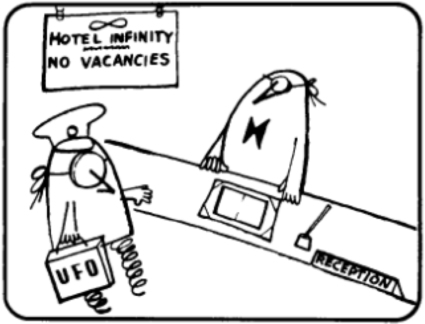
An African native and Prof. G. Cantor comparing the numbers beyond their counting ability.

The paradoxes of the infinity



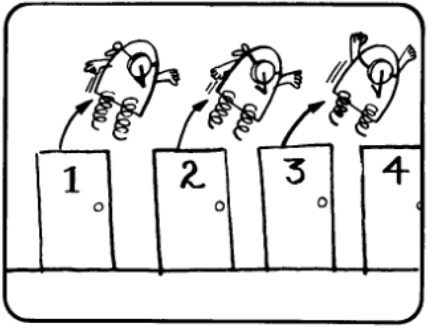
Before Dr. Zeta left, he told a fantastic story.
Dr. Zeta: Hotel Infinity is an enormous hotel at the center of our galaxy. It has an infinite number of rooms that extend through a black hole into a higher dimension. The room numbers start at 1 and go on forever.

The paradoxes of the infinity



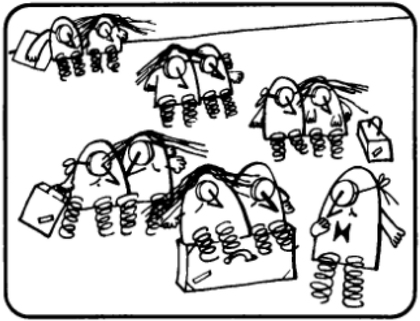
Dr. Zeta: One day, when every room was occupied, a UFO pilot, on his way to another galaxy, arrived.

The paradoxes of the infinity



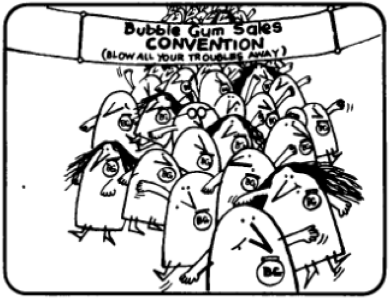
Dr. Zeta: Even though there was no vacancy, the hotel manager found a room for the pilot. He just moved the occupants of each room to the room with a number that was one higher. This left Room 1 vacant for the pilot.

The paradoxes of the infinity



Dr. Zeta: The next day, five couples on their honeymoons showed up. Could Hotel Infinity take care of them? Yes, the manager simply moved everybody to a room with a number that was five higher. This left rooms 1 through 5 vacant for the five couples.

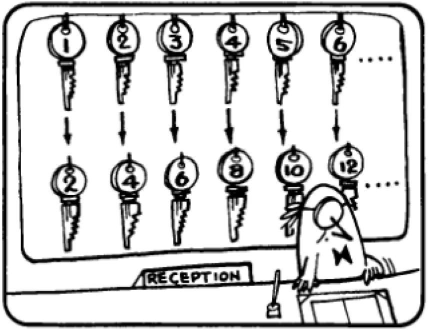
The paradoxes of the infinity



Dr. Zeta: On the weekend an *infinite* number of bubble gum salespeople came to the hotel for a convention.

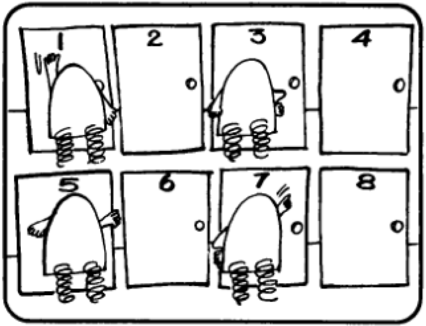
Herman: I can understand how Hotel Infinity could take care of any *finite* number of new arrivals. But how could it find room for an *infinite* number?

The paradoxes of the infinity



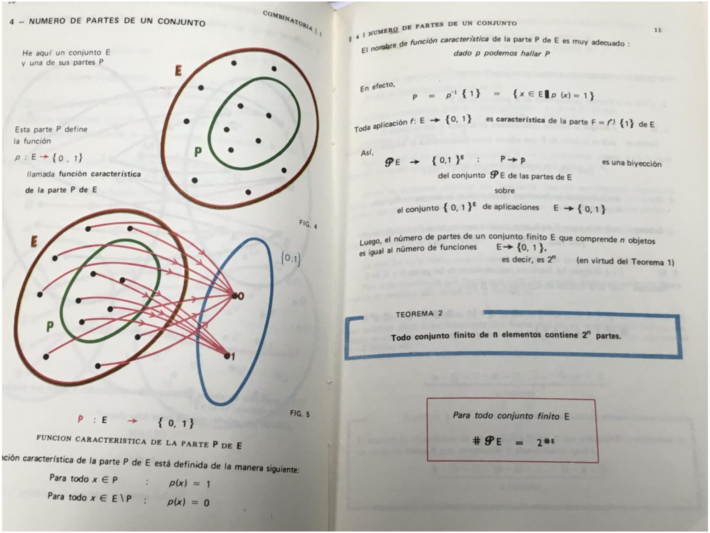
Dr. Zeta: Easily, my dear Herman. The manager just moved everyone to a room with a number *twice* as large as before.

The paradoxes of the infinity



Herman: Of course! That put everybody in a room with an *even* number. This left all the odd-number rooms—an infinity of them—vacant for the bubble gummers!

Set Theory



4 – NUMERO DE PARTES DE UN CONJUNTO

COMBINATORIA | 1

He aquí un conjunto E
y una de sus partes P

Esta parte P define
la función

$p : E \to \{0, 1\}$

llamada **función característica**

de la parte P de E

FIG. 4

FIG. 5

p : E → { 0, 1 }

FUNCION CARACTERISTICA DE LA PARTE P DE E

...ción característica de la parte P de E está definida de la manera siguiente:

Para todo $x \in P$: $p(x) = 1$

Para todo $x \in E \setminus P$: $p(x) = 0$

§ 4 | NUMERO DE PARTES DE UN CONJUNTO 11

El nombre de *función característica* de la parte P de E es muy adecuado :

dado p podemos hallar P

En efecto,

$$P = p^{-1}\{1\} = \{x \in E \mid p(x) = 1\}$$

Toda aplicación f: E → {0, 1} es **característica** de la parte $F = f^{-1}\{1\}$ de E

Así,

$\mathcal{P}E \to \{0,1\}^E$: $P \to p$ es una biyección

del conjunto $\mathcal{P}E$ de las partes de E

sobre

el conjunto $\{0,1\}^E$ de aplicaciones E → {0, 1}

Luego, el número de partes de un conjunto finito E que comprende n objetos es igual al número de funciones E → {0, 1},

es decir, es 2^n (en virtud del Teorema 1)

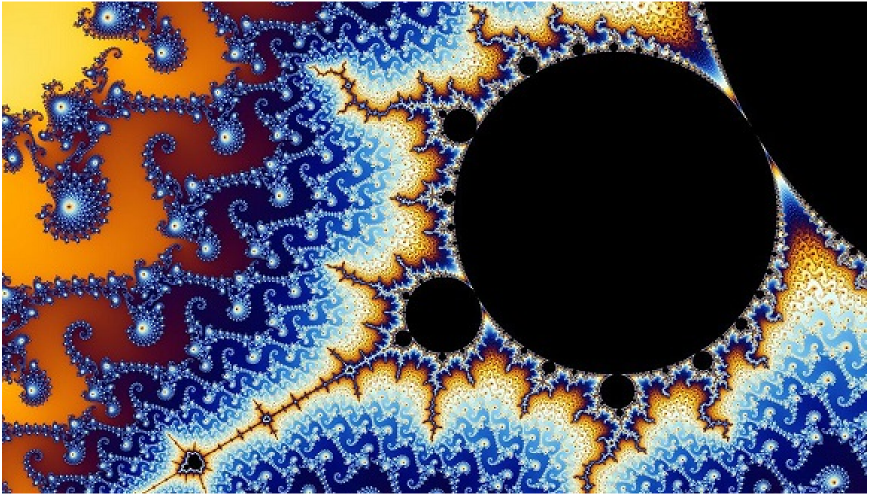
TEOREMA 2

Todo conjunto finito de n elementos contiene 2^n partes.

Para todo conjunto finito E

$$\# \mathcal{P}E = 2^{\# E}$$

Set Theory: point sets in the plane



Measure Theory: Henry Lebesgue

Measure Theory: integration

LEÇONS

SUR L'INTÉGRATION

ET LA

RECHERCHE DES FONCTIONS PRIMITIVES,

PROFESSÉES AU COLLÈGE DE FRANCE

PAR

Henri LEBESGUE,

MAÎTRE DE CONFÉRENCES A LA FACULTÉ DES SCIENCES DE RENNES.

Measure Theory: integration

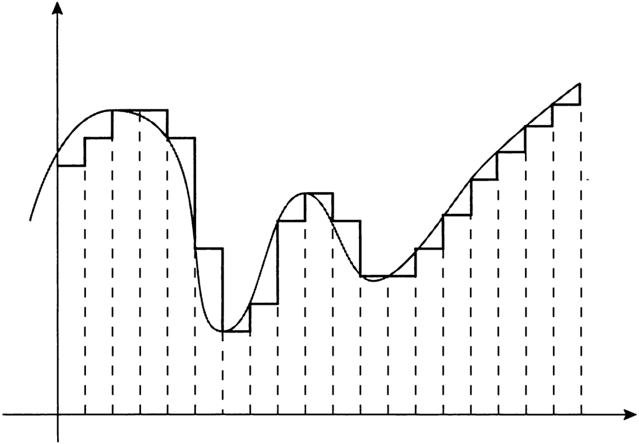


Figure 5a
Méthode de Riemann

Measure Theory: integration

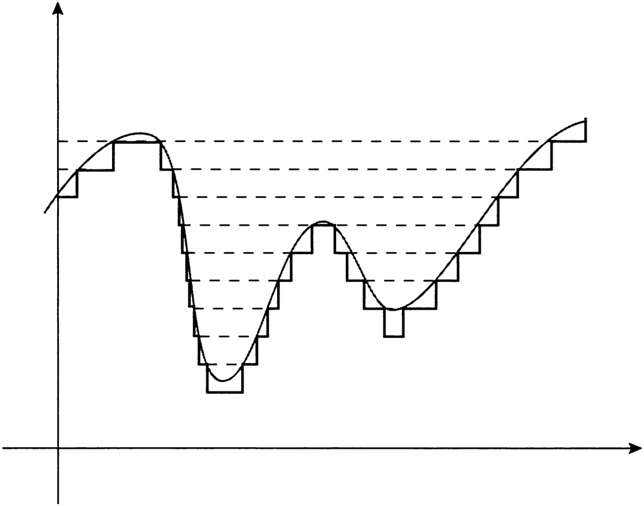
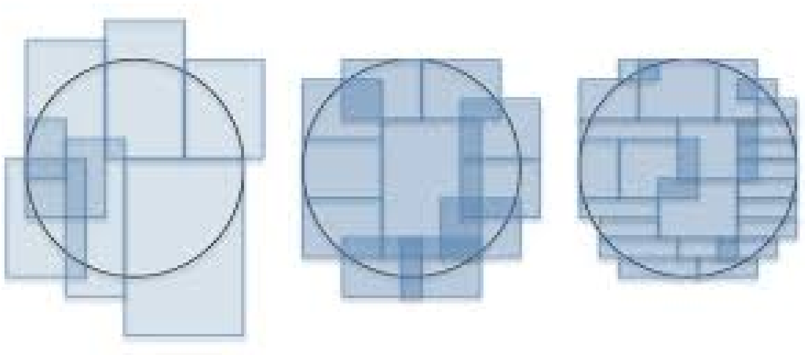


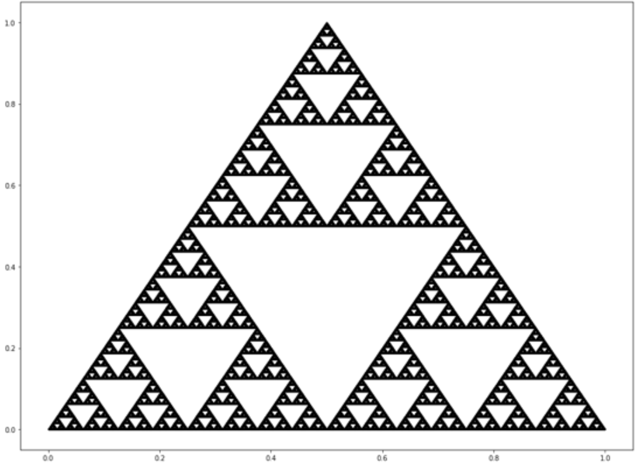
Figure 5b
Méthode de Lebesgue

Measure Theory: the outer measure



The only elementary fact we requiere is the additivity of the area for rectangles whose sides are parallel to the axes.

Measure Theory: the generality



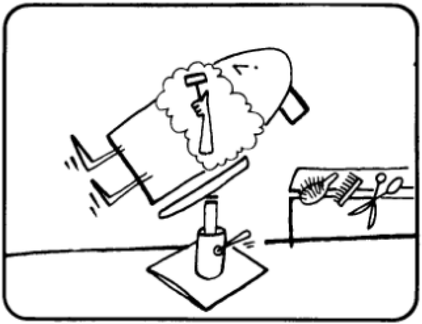
All the "definable sets" are measurable (in the sense of Lebesgue)... Is there a non-measurable set? We need to go a little deeper in Set Theory in order to answer the question.

Set Theory meets Logic

But Set Theory is not as simple at it seems.

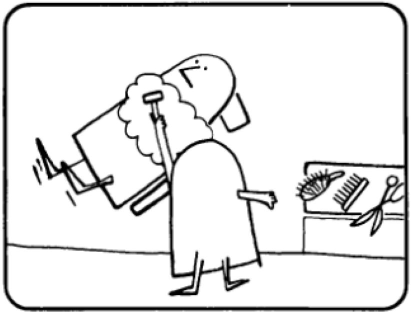
The famous barber paradox was proposed by Bertrand Russell. If a barber has the sign at the left in his window, who shaves the barber?

Set Theory meets Logic



If he shaves himself, then he belongs to the set of men who shave themselves. But his sign says he *never* shaves anyone in this set. Therefore he *cannot* shave himself.

Set Theory meets Logic

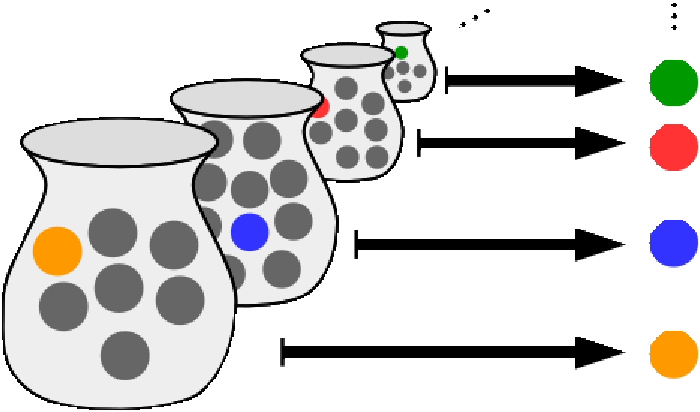


If someone else shaves the barber, then he's a man who doesn't shave himself. But his sign says that he *does* shave *all* such men. Therefore no one else can shave the barber. It seems as if *nobody* can shave the barber!

This Paradox was adapted by Bertrand Russell and it almost destroy Set Theory at its very beginning. . . The foundations of Set Theory should be taken very seriously too.

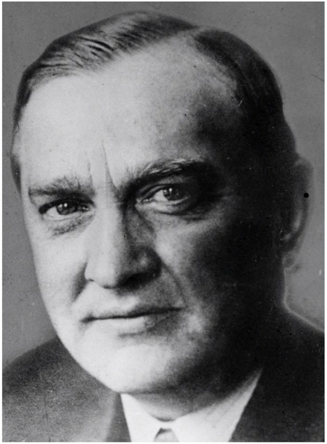
The Axiom of Choice

It is possible to “build” a non-measurable set using a somehow “polemic axiom” from Set Theory



Now, Mathematics as a whole, is founded on Set Theory and Logic.

The Banach-Tarski paradox



The Banach-Tarski paradox

The Axiom of Choice has many nice consequences in Mathematics. For instance, the existence of a finitely additive "measure of area" defined on all the subsets of the plane. But the Axiom of Choice is also the key for one of the weirdest mathematical phenomenons.

Banach and Tarski proved that a (3-dimensional) euclidean ball can be decomposed into finitely many parts in such way that rearranged produce two balls alike to the original one.

Gödel's incopleteness

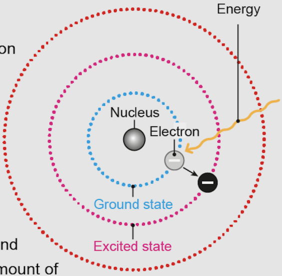
Is the Axiom of Choice the last brick for the foundation of Mathematics?

The Spectral Gap

There is a difference between Real World and Mathematics: in the Real World we have the Conservation of Mass Law; in the mathematical universe we have the Banach-Tarski paradox. . . Are we sure that the Axiom of Choice doesn't mess with the Real World?

The Spectral Gap

The authors' mathematical proof took on the question of the "spectral gap"—the jump in energy between the ground state and first excited state of a material. When we think of energy states, we tend to think of electrons in atoms, which can jump up and down between energy levels. Whereas in atoms there is always a gap between such levels, in larger materials made of many atoms, there is sometimes no distance between the ground state and the first excited state: even the smallest possible amount of energy will be enough to push the material up an energy level. Such materials are called "gapless." The authors proved that it will never be possible to determine whether all materials are gapped or gapless.



The Spectral Gap

Undecidability of the Spectral Gap

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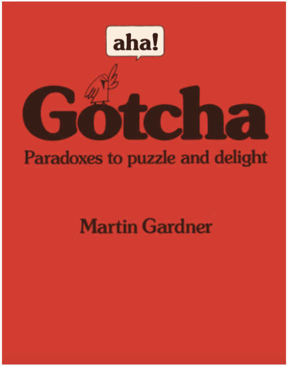
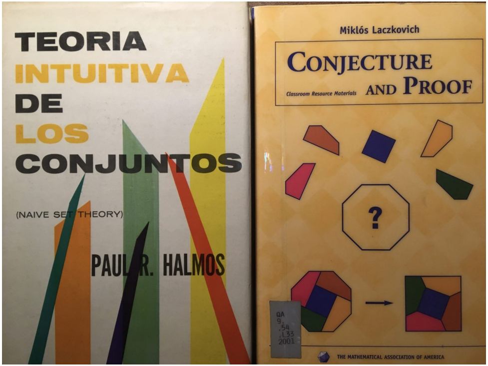
Some books


aha!
Gotcha
Paradoxes to puzzle and delight
Martin Gardner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